Minimal single linear functional observers for linear systems☆

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ABSTRACT

A constructive procedure to design a single linear functional observer for a time-invariant linear system is given. The proposed procedure is simple and is not based on the solution of a Sylvester equation or on the use of canonical state space forms. Both stable observers or fixed poles observers problems are considered for minimality.

1. Introduction

Since Luenberger's works (Luenberger, 1963, 1964, 1966) a significant amount of research has been devoted to the problem of observing a linear functional

$$v(t) = Lx(t), \tag{1}$$

where L is a constant full row rank $(l \times n)$ matrix, and, for every time t in $\mathbb{R}^+$, $x(t)$ is the n-dimensional state vector of the state space system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2}$$

where $u(t)$ is the p-dimensional control, and $y(t)$ is the m-dimensional measure. $A(n \times n)$, $B(n \times p)$ and $C(m \times n)$ are constant matrices. For a survey of the main results see for instance (Aldeen & Trinh, 1999; O'Reilly, 1983; Trinh & Fernando, 2007; Tsui, 1998) and the references therein.

The observation of $v(t)$ can be carried out with the design of the Luenberger observer

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gu(t) + Hy(t), \\ w(t) &= Pz(t) + Vy(t), \end{aligned} \tag{3}$$

where $z(t)$ is the q-dimensional state vector. Constant matrices F, G, H, P and V are determined such that

$$\lim_{t\to\infty} (v(t) - w(t)) = 0.$$

This asymptotic tracking is ensured if F is a Hurwitz matrix. Namely, if all the eigenvalues of F are such that their real part is negative.

We know from Fortmann and Williamson (1972) that the linear functional observer (3) exists if and only if there exists a $(q \times n)$ matrix T such that:

$$G = TB,$$

$$TA - FT = HC, \tag{4}$$

$$L = PT + VC, \tag{5}$$

$$F \text{ is a Hurwitz matrix.} \tag{6}$$

Notice that a rigorous proof of this result has been established in Fuhrmann and Helmke (2001) for the case $V = 0$. According to the value of q, we distinguish several observers:

- $q = n$: the Kalman observer;
- $q = n - m$: the reduced-order observer or Cumming–Gopinath observer (Cumming, 1969; Gopinath, 1971);
- $l < q < n - m$ and q is such that no observer of $v(t)$ with an order less than q exists: the minimal-order observer or minimal observer;
- $l = q$: the minimum-order observer or minimum observer. With (Roman & Bullock, 1975; Sirisena, 1979), we know that l is a lower bound for the order of the observer (3).

Until now the direct design of a minimal observer of a given linear functional is an open question. Since (Fortmann & Williamson, 1972), design schemes have been proposed to reduce the order of the observer (3) with respect to the reduced-order observer. Mainly, these designs are based on the determination of the matrices T and F such that the Sylvester equation (4) is fulfilled (Trinh, Nahavandi, & Tran, 2008; Tsui, 2004). Unfortunately, the problem rests in satisfying conditions (4)–(6) with the dimension

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of F minimal. Moreover the distinction between the fixed-pole observer problem where the poles are specified at the outset and the stable observer problem where the poles are permitted to lie anywhere in the left half-plane is not well defined. Indeed, in several cases, necessary and sufficient existence conditions for a candidate observer to be minimal are obtained for the fixed-pole observer problem only.

Apart from results on the multifunctional case ($l > 1$), for simplification purpose several authors have considered the observation of a single linear functional ($l = 1$). Luenberger has shown in Luenberger (1966) that any specified single linear functional of the state vector may be obtained by means of an observer of order $(\nu - 1)$, with arbitrary dynamics, where ν is the observability index of the system. Namely, ν is the smallest integer for which the matrix

$$\mathcal{O}_{(A,C),\nu} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu-1} \end{bmatrix}$$

has rank n. In Murdoch (1973), an effective design procedure is proposed to design an observer of order $(\nu-1)$, with arbitrary dynamics. Some constraints are imposed on the choice of the observer poles or on structures of some matrices. In addition, a method to apply this result to the multifunctional case has been proposed in Murdoch (1974). Some years later, based on simplifications brought about by considering (Roman & Bullock, 1975)

$$C = \begin{bmatrix} 0_{m\times(n-m)} & I_m \end{bmatrix},$$

a necessary and sufficient condition is proposed in Gupta, Fairmann, and Hinamoto (1981) for the design of a single functional observer in a fixed-pole problem. Another condition for the existence of a q-order observer can be found in Kondo and Sunaga (1986) which can be related to the Fortmann–Williamson condition (Fortmann & Williamson, 1972). In the particular case of a single functional, the minimum order observer is a one-order filter and may not exist. Some conditions for the existence of a second order observer are given in Trinh and Zhang (2005). Based on the Duan procedure to solve a Sylvester equation (Duan, 1993), the design of a general order observer for a single functional is proposed in Trinh, Tran, and Nahavandi (2006). To apply this method, which is extended to the multifunctional observer in Trinh et al. (2008), the poles of the observer have to be distinct and must be fixed at the outset. Nevertheless, the proposed procedures are based on the solution of the Sylvester equation and some numerical algorithms exist to increase the numerical robustness in the design (Datta, 2004; Datta & Sarkissian, 2000).

The present paper describes a direct and iterative procedure to get a minimal order for a single linear functional observer. On the one hand the word *direct* means that the design method is not based on the solution of the Sylvester equation (4). It has been underlined in Tsui (1998), that the calculus of the matrix T is not a necessary step. This point is a specific feature of the procedure we propose. On the other hand the word iterative indicates that we test an increasing sequence for the orders of the observers to obtain minimality. Moreover, the procedure points out if we face to the stable observer case or if some poles can be fixed at the outset.

The paper is organised as follows. In the first place, a necessary and sufficient condition is outlined for the existence of a single functional observer. From these conditions, in the second section a design method for the observer is proposed. An example illustrates the procedure and points out that the minimal order depends on constraints on the poles of the observer.

Notice that, in all the following, we suppose

$$\operatorname{rank}\begin{bmatrix} C \\ L \end{bmatrix} = m + 1.$$

This condition rules out the design of an obvious nondynamic observer and can be verified easily.

2. Minimal observers existence condition

Let us define q as the smallest integer such that

$$\operatorname{rank} \Sigma_q = \operatorname{rank}\begin{bmatrix} \Sigma_q \\ LA^q \end{bmatrix}, \tag{7}$$

with

$$\Sigma_q = \begin{bmatrix} C \\ L \\ CA \\ LA \\ \vdots \\ CA^{q-1} \\ LA^{q-1} \\ CA^q \end{bmatrix}.$$

After q derivatives of $v(t) = Lx(t)$ we obtain

$$v^{(q)}(t) = LA^q x(t) + \sum_{i=0}^{q-1} LA^i Bu^{(q-1-i)}(t). \tag{8}$$

From (7) there exist $\Gamma_{(i)}$, for $i = 0$ to q, and Λ_i, for $i = 0$ to $q-1$, such that

$$LA^q = \sum_{i=0}^{q} \Gamma_{(i)} CA^i + \sum_{i=0}^{q-1} \Lambda_i LA^i. \tag{9}$$

Thus (8) can be written

$$v^{(q)}(t) = \sum_{i=0}^{q} \Gamma_{(i)} CA^i x(t) + \sum_{i=0}^{q-1} \Lambda_i LA^i x(t) + \sum_{i=0}^{q-1} LA^i Bu^{(q-1-i)}(t).$$

To eliminate the state $x(t)$ we have the equalities

$$\begin{aligned}
Lx(t) &= v(t), \\
LAx(t) &= \dot{v}(t) - LBu(t), \\
&\vdots \\
LA^{(q-1)}x(t) &= v^{(q-1)} - \sum_{i=0}^{q-2} LA^i Bu^{(q-2-i)}(t), \\
Cx(t) &= y(t), \\
CAx(t) &= \dot{y}(t) - CBu(t), \\
CA^2 x(t) &= \ddot{y}(t) - CABu(t) - CB\dot{u}(t), \\
&\vdots \\
CA^{(q)}x(t) &= y^{(q)} - \sum_{i=0}^{q-1} CA^i Bu^{(q-1-i)}(t).
\end{aligned}$$

It yields

$$\begin{aligned}
v^{(q)}(t) = {} & \Gamma_{(0)} y(t) + \sum_{i=1}^{q} \Gamma_{(i)} \left[y^{(i)} - \sum_{j=0}^{i-1} CA^j Bu^{(i-1-j)}(t) \right] \\
& + \Lambda_0 v(t) + \sum_{i=1}^{q-1} \Lambda_i \left[v^{(i)} - \sum_{j=0}^{i-1} LA^j Bu^{(i-1-j)}(t) \right] \\
& + \sum_{i=0}^{q-1} LA^i Bu^{(q-1-i)}(t), \\
= {} & \sum_{i=0}^{q} \Gamma_{(i)} y^{(i)} + \sum_{i=0}^{q-1} \Lambda_i v^{(i)} + \sum_{i=0}^{q-1} \Phi_i u^{(i)}(t),
\end{aligned} \tag{10}$$

where for $i = 0$ to $q-2$

$$\Phi_i = \left[LA^{q-1-i} - \sum_{j=i+1}^{q} \Gamma_{(j)} CA^{j-i-1} - \sum_{j=i+1}^{q-1} \Lambda_j LA^{j-i-1} \right] B,$$

and

$$\Phi_{q-1} = \left[L - \Gamma_{(q)} C \right] B.$$

The input–output differential equation (10) can be realized as the q-order state space observable system

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} 0 & & & \Lambda_0 \\ 1 & \ddots & & \Lambda_1 \\ & \ddots & 0 & \vdots \\ & & 1 & \Lambda_{q-1} \end{bmatrix} z(t) + \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_{q-1} \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} \Gamma_{(0)} + \Lambda_0 \Gamma_{(q)} \\ \Gamma_{(1)} + \Lambda_1 \Gamma_{(q)} \\ \vdots \\ \Gamma_{(q-1)} + \Lambda_{q-1} \Gamma_{(q)} \end{bmatrix} y(t), \\ v(t) &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} z(t) + \Gamma_{(q)} y(t). \end{aligned} \tag{11}$$

Theorem 1. *Let us define q as the smallest integer such that there exist Γ_i, for $i = 0$ to q, and Λ_i, for $i = 0$ to $q-1$, such that*

$$LA^q = \sum_{i=0}^{q} \Gamma_{(i)} CA^i + \sum_{i=0}^{q-1} \Lambda_i LA^i,$$

and the matrix

$$F = \begin{bmatrix} 0 & & & \Lambda_0 \\ 1 & \ddots & & \Lambda_1 \\ & \ddots & 0 & \vdots \\ & & 1 & \Lambda_{q-1} \end{bmatrix}$$

is a Hurwitz matrix. Then the q-order Luenberger observer (11) is a minimal observer of the single linear functional (1) for the system (2).

Proof. As $G = TB$, the form of the Φ_i leads to the matrix

$$T = \begin{bmatrix} LA^{q-1} - \Gamma_{(1)} C - \cdots - \Gamma_{(q)} CA^{q-1} \\ \quad -\Lambda_1 L - \cdots - \Lambda_{q-1} LA^{q-1} \\ \vdots \\ LA - \Gamma_{(q-1)} C - \Gamma_{(q)} CA - \Lambda_{q-1} L \\ L - \Gamma_{(q)} C \end{bmatrix}.$$

Somecalculations point out that (5) and (4) are verified. When F is a Hurwitz matrix, the necessary and sufficient conditions for the existence of a single functional observer of the single linear functional (1) for the system (2) are fulfilled. To prove minimality let us consider that there exists a p-observer solving the same problem with $p < q$. Then there exist matrices such that

$$v^{(p)}(t) = \sum_{i=0}^{p} R_i y^{(i)} + \sum_{i=0}^{p-1} S_i v^{(i)} + \sum_{i=0}^{p-1} T_i u^{(i)}(t).$$

Taking into account

$$\begin{aligned} v(t) &= Lx(t), \\ \dot{v}(t) &= LAx(t) + LBu(t), \\ &\vdots \\ v^{(p)} &= LA^{(p)} x(t) + \sum_{i=0}^{p-1} LA^i Bu^{(p-1-i)}(t), \\ y(t) &= Cx(t), \\ \dot{y}(t) &= CAx(t) + CBu(t), \\ &\vdots \\ y^{(p)} &= CA^p x(t) + \sum_{i=0}^{p-1} CA^i Bu^{(p-1-i)}(t), \end{aligned}$$

and supposing, for simplicity sake, that $u(t)$ vanishes for every t, we get

$$LA^p x(t) = \sum_{i=0}^{p} R_i CA^i x(t) + \sum_{i=0}^{p-1} S_i LA^i x(t).$$

This relation must be fulfilled for all solutions of $\dot{x}(t) = Ax(t)$. Consequently LA^p is linearly dependent on the rows CA^i and LA^i. Let us suppose that the matrix

$$\begin{bmatrix} 0 & & & S_0 \\ 1 & \ddots & & S_1 \\ & \ddots & 0 & \vdots \\ & & 1 & S_{p-1} \end{bmatrix}$$

is a Hurwitz matrix. This point is a contradiction, because q is the smallest integer such that the writing (9) exists. □

For completeness, let us describe the existence condition of a minimum order observer for a single linear functional. In this case, it can readily be seen that the Theorem 1 gives: a one-order observer for $Lx(t)$ exists if and only if we can write

$$LA = \Gamma_{(0)} C + \Lambda_0 L + \Gamma_{(1)} CA, \tag{12}$$

where Λ_0 is strictly negative. It can been shown that this condition is equivalent to the condition established in Darouach (2000).

3. Design procedure and pole placement

3.1. Design procedure

We develop in this section the procedure to design a q-order single functional observer when the conditions for the existence of a p-order single functional observer with $1 \le p < q$ are not fulfilled. In order to examine if some of the poles of the observer can be fixed at the outset we introduce the following partitions

$$C = C_0 = \begin{bmatrix} C_1 \\ C_1^* \end{bmatrix} = \begin{bmatrix} C_2 \\ C_2^* \end{bmatrix} = \cdots = \begin{bmatrix} C_q \\ C_q^* \end{bmatrix}, \tag{13}$$

where q is defined by (9) and for $i = 1$ to q, the rows of C_i and C_i^* are such that

- $C_i A^i$ is linearly independent of $C, L, CA, LA, \ldots, CA^{i-1}$;
- $C_i^* A^i$ is linearly dependent of $C, L, CA, LA, \ldots, CA^{i-1}$.

It is obvious that the previous partitions of C possibly necessitates a permutation in the measure variables. Let us denote π_i, $i = 1$ to q, the number of rows in C_i^*. We have $0 \le \pi_1 \le \pi_2 \le \cdots \le \pi_q$.

Associated with the partitions (13) we have the partitions, for $i = 1$ to q, $\Gamma_{(i)} = \left[\Gamma_i \quad \Gamma_i^* \right]$, where Γ_i^* is a $(l \times \pi_i)$ matrix. From (9), it yields

$$LA^q = \sum_{i=0}^{q} \Gamma_i C_i A^i + \sum_{i=1}^{q} \Gamma_i^* C_i^* A^i + \sum_{i=0}^{q-1} \Lambda_i LA^i. \tag{14}$$

The matrix

$$\Sigma_q^* = \begin{bmatrix} C \\ L \\ C_1 A \\ LA \\ \vdots \\ C_{q-1}A^{q-1} \\ LA^{q-1} \\ C_q A^q \end{bmatrix},$$

is a full-row rank matrix with

$$\text{rank } \Sigma_q^* = (q+1)m + q - \sum_{i=1}^{q} \pi_i.$$

Thus, the matrices Π_i and Δ_i defined by

$$LA^q = \sum_{i=0}^{q} \Pi_i C_i A^i + \sum_{i=0}^{q-1} \Delta_i LA^i, \tag{15}$$

and the matrices $\Gamma_{i,j}$ and $\Lambda_{i,j}$ defined, for $i = 1$ to q, by

$$C_i^* A^i = \sum_{j=0}^{i} \Gamma_{i,j} C_j A^j + \sum_{j=0}^{i-1} \Lambda_{i,j} LA^j, \tag{16}$$

are unique. Taking into account (16) in (14) yields

$$LA^q = \left[\Gamma_0 + \sum_{i=1}^{q} \Gamma_i^* \Gamma_{i,0}\right] C_0 + \sum_{i=1}^{q} \left[\Gamma_i + \sum_{j=i}^{q} \Gamma_j^* \Gamma_{j,i}\right] C_i A^i + \sum_{i=0}^{q-1} \left[\Lambda_i + \sum_{j=i+1}^{q} \Gamma_j^* \Lambda_{j,i}\right] LA^i.$$

From the unique property in (15) we deduce the coefficients for the minimal observer

$$\Gamma_0 = \Pi_0 - \sum_{i=1}^{q} \Gamma_i^* \Gamma_{i,0},$$

$$\text{for } i = 1 \text{ to } q, \quad \Gamma_i = \Pi_i - \sum_{j=i}^{q} \Gamma_j^* \Gamma_{j,i},$$

$$\text{for } i = 0 \text{ to } q-1, \quad \Lambda_i = \Delta_i - \sum_{j=i+1}^{q} \Gamma_j^* \Lambda_{j,i},$$

where the Γ_i^* are design parameters. Specifically, the characteristic polynomial of the matrix

$$F = \begin{bmatrix} 0 & & & \Lambda_0 \\ 1 & \ddots & & \Lambda_1 \\ & \ddots & 0 & \vdots \\ & & 1 & \Lambda_{q-1} \end{bmatrix}$$

is given by

$$p_F(\lambda) = \lambda^q - \sum_{i=0}^{q-1} \left[\Delta_i - \sum_{j=i+1}^{q} \Gamma_j^* \Lambda_{j,i}\right] \lambda^i.$$

The total number of design parameters to obtain stable poles for the observer or to solve the fixed-pole observer problem is $\sigma = \sum_{i=1}^{q} \pi_i$. When $\sigma \geq q$, all the poles can be fixed at the outset. On the opposite side, when $\sigma < q$, σ indicates the number of poles which can be fixed at the outset. In this case, when the stable observer problem cannot be solved, the solution consists in increasing the order of the observer. This step is performed by taking the derivative of $\upsilon^{(q)}$. Namely, the procedure we have detailed in this section is applied on

$$\upsilon^{(q+1)}(t) = LA^{q+1}x(t) + \sum_{i=0}^{q} LA^i Bu^{(q-i)}(t).$$

We can remark that the decomposition (9) yields

$$\begin{aligned} LA^{q+1} &= LA^q A = \left[\sum_{i=0}^{q} \Gamma_{(i)} CA^i + \sum_{i=0}^{q-1} \Lambda_i LA^i\right] A, \\ &= \sum_{i=1}^{q+1} \Gamma_{(i-1)} CA^i + \sum_{i=1}^{q} \Lambda_{i-1} LA^i. \end{aligned}$$

Thus this decomposition is immediate and the previous method can be used to design a minimal observer with some constraints on the poles.

3.2. Illustrative example

With (Trinh et al., 2006), let us consider the system (2) and the single functional (1) defined by

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 & -2 \\ 0 & -5 & 3 & 4 & 0 \\ 1 & 1 & -8 & 3 & 0 \\ -4 & 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{17}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$L = \begin{bmatrix} 1 & 14 & 42 & 79 & 2 \end{bmatrix}.$$

The following steps illustrate the design procedure of minimal observers.

(1) Test for the minimum observer. As

$$CA = \begin{bmatrix} -1 & 0 & 0 & 1 & -2 \end{bmatrix},$$
$$LA = \begin{bmatrix} -275 & -28 & -136 & -289 & -4 \end{bmatrix},$$

we obtain rank $(\Sigma_1) = 3$ and

$$\text{rank} \begin{bmatrix} \Sigma_1 \\ LA \end{bmatrix} = 4.$$

Thus a first-order minimum observer cannot be designed.

(2) Tests for a second-order observer. As

$$CA^2 = \begin{bmatrix} -3 & 0 & 2 & -9 & 4 \end{bmatrix},$$
$$LA^2 = \begin{bmatrix} 1295 & 4 & 426 & 935 & 554 \end{bmatrix},$$

we get rank $(\Sigma_2) = 5$ and

$$\text{rank} \begin{bmatrix} \Sigma_2 \\ LA^2 \end{bmatrix} = 5.$$

As

$$LA^2 \Sigma_2^{-1} = \begin{bmatrix} -1343.7 & -16.6 & -301 & -8.4 & -12.1 \end{bmatrix},$$

we deduce $\Lambda_0 = -16.6$ and $\Lambda_1 = -8.4$. It yields

$$F = \begin{bmatrix} 0 & -16.6 \\ 1 & -8.4 \end{bmatrix}.$$

The eigenvalues of F are $\{-3.12, -5.22\}$. Thus a minimal second-order observer can be designed.

(3) Design of the minimal second-order observer. From $LA^2\Sigma_2^{-1}$ we get $\Gamma_{(0)} = -1343.7$, $\Gamma_{(1)} = -301$ and $\Gamma_{(2)} = -12.1$. From (11) we obtain

$$G = \begin{bmatrix} 22.29 & 90 & 218 & 389 & -11.42 \\ 13.14 & 14 & 42 & 79 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -11.42 \\ 2 \end{bmatrix},$$

$$H = \begin{bmatrix} -1142.5 \\ -198.7 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad V = -12.1.$$

The design of this observer is finished and the procedure can be stopped. Nevertheless the poles are fixed. If the poles $\{-3.12, -5.22\}$ are not acceptable we have to augment the order of the observer. In the next step we tackle this point.

(4) Design of a minimal third-order observer with partially fixed poles. In order to illustrate the design procedure we detail some calculations here. From

$$CA^3 = \begin{bmatrix} \Gamma_{20} & \Lambda_{20} & \Gamma_{21} & \Lambda_{21} & \Gamma_{22} \end{bmatrix} \Sigma_2,$$

$$LA^2 = \begin{bmatrix} \Pi_0 & \Delta_0 & \Pi_1 & \Delta_1 & \Pi_2 \end{bmatrix} \Sigma_2,$$

we deduce, on the one hand

$$\begin{aligned} LA^3 &= LA^2A = \Pi_0 CA + \Delta_0 LA + \Pi_1 CA^2 + \Delta LA^2 + \Pi_2 CA^3, \\ &= \Pi_0 CA + \Delta_0 LA + \Pi_1 CA^2 \\ &\quad + \Delta_1 \begin{bmatrix} \Pi_0 & \Delta_0 & \Pi_1 & \Delta_1 & \Pi_2 \end{bmatrix} \Sigma_2 \\ &\quad + \Pi_2 \begin{bmatrix} \Gamma_{20} & \Lambda_{20} & \Gamma_{21} & \Lambda_{21} & \Gamma_{22} \end{bmatrix} \Sigma_2, \\ &= [\Delta_1\Pi_0 + \Pi_2\Gamma_{20} \quad \Delta_1\Delta_0 + \Pi_2\Lambda_{20} \quad \Pi_0 + \Delta_1\Pi_1 + \Pi_2\Gamma_{20} \\ &\quad\ \Delta_0 + \Delta_1\Delta_1 + \Pi_2\Lambda_{21} \quad \Pi_1 + \Delta_1\Pi_2 + \Pi_2\Gamma_{22}]\Sigma_2, \end{aligned}$$

and on the other hand

$$\begin{aligned} LA^3 &= \Gamma_0 C + \Lambda_0 L + \Gamma_{(1)} CA + \Lambda_1 LA + \Gamma_{(2)} CA^2 + \Lambda_2 LA^2 + \Gamma_{(3)} CA^3, \\ &= [\Gamma_0 + \Lambda_2\Pi_0 + \Gamma_{(3)}\Gamma_{20} \quad \Lambda_0 + \Lambda_2\Delta_0 + \Gamma_{(3)}\Lambda_{20} \\ &\quad\ \Gamma_{(1)} + \Lambda_2\Pi_1 + \Gamma_{(3)}\Gamma_{21} \quad \Lambda_1 + \Lambda_2\Delta_1 + \Gamma_{(3)}\Lambda_{21} \\ &\quad\ \Gamma_{(2)} + \Lambda_2\Pi_2 + \Gamma_{(3)}\Gamma_{22}]\Sigma_2, \end{aligned}$$

where Λ_2 and $\Gamma_{(3)}$ are two design parameters. It yields

$$\Gamma_0 = \Delta_1\Pi_0 + \Pi_2\Gamma_{20} - \Lambda_2\Pi_0 - \Gamma_{(3)}\Gamma_{20},$$

$$\Gamma_{(1)} = \Pi_0 + \Delta_1\Pi_1 + \Pi_2\Gamma_{21} - \Lambda_2\Pi_1 - \Gamma_{(3)}\Gamma_{21},$$

$$\Gamma_{(2)} = \Pi_1 + \Delta_1\Pi_2 + \Pi_2\Gamma_{22} - \Lambda_2\Pi_2 - \Gamma_{(3)}\Gamma_{22},$$

and

$$\Lambda_0 = \Delta_1\Delta_0 + \Pi_2\Lambda_{20} - \Lambda_2\Delta_0 - \Gamma_{(3)}\Lambda_{20},$$

$$\Lambda_1 = \Delta_0 + \Delta_1\Delta_1 + \Pi_2\Lambda_{21} - \Lambda_2\Delta_1 - \Gamma_{(3)}\Lambda_{21}.$$

When Λ_2 and $\Gamma_{(3)}$ are chosen, these five parameters are known and we can implement the observer (11) for $q = 3$. The poles of the matrix F are the roots of the characteristic polynomial $p_F(\lambda) = \lambda^3 - \Lambda_2\lambda^2 - \Lambda_1\lambda - \Lambda_0$ which depends on the parameters Λ_2 and $\Gamma_{(3)}$. In our example

$$CA^3 = \begin{bmatrix} 41 & 2 & -34 & 61 & 2 \end{bmatrix},$$

$$LA^3 = \begin{bmatrix} -4609 & 406 & -1526 & -2467 & -3144 \end{bmatrix}.$$

With $CA^3\Sigma_2^{-1}$ and $LA^3\Sigma_2^{-1}$ we obtain the values

$$\Gamma_{20} = 55.14, \qquad \Gamma_{21} = -26, \qquad \Gamma_{22} = -12.57,$$

$$\Lambda_{20} = 0.71, \qquad \Lambda_{21} = 0.29,$$

$$\Delta_1\Pi_0 + \Pi_2\Gamma_{20} = 10\,656,$$

$$\Pi_0 + \Delta_1\Pi_1 + \Pi_2\Gamma_{21} = 1509,$$

$$\Pi_1 + \Delta_1\Pi_2 + \Pi_2\Gamma_{22} = -46,$$

$$\Delta_1\Delta_0 + \Pi_2\Lambda_{20} = 131,$$

$$\Delta_0 + \Delta_1\Delta_1 + \Pi_2\Lambda_{21} = 51.$$

As $\Delta_1 = -8.4$ and $\Delta_0 = -16.6$, it can be read

$$\Lambda_0 = 131 + 16.6\Lambda_2 - 0.71\Gamma_{(3)},$$

$$\Lambda_1 = 51 + 8.4\Lambda_2 - 0.29\Gamma_{(3)},$$

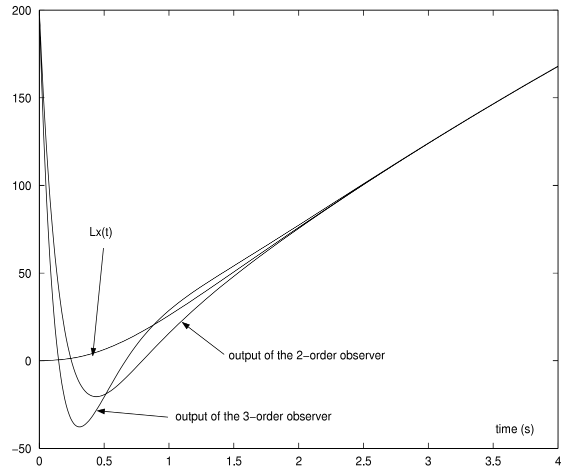

Fig. 1. Simulation results for the implementation of the reduced order observers for (17).

and the polynomial

$$\begin{aligned} p_F(\lambda) &= \lambda^3 - 51\lambda - 131 - \Lambda_2(\lambda^2 + 8.4\lambda + 16.6) \\ &\quad + \Gamma_{(3)}(0.29\lambda + 0.71). \end{aligned}$$

In order to compare it with (Trinh et al., 2006) where the roots of $p_F(\lambda)$ are $\{-3, -4, -5\}$ we get $\Lambda_2 = -12$, $\Lambda_1 = -47$ and $\Lambda_0 = -60$. These equalities are consistent and yield $\Gamma_{(3)} = -11$. Despite this result it is not obvious to give three poles at the outset such that the constraints are satisfied. In order to compare with (Trinh et al., 2006) we can fix $\Lambda_2 = -12$. We are then led to

$$p_F(\lambda) = \lambda^3 + 12\lambda^2 + 50.14\lambda + 67.86 + \Gamma_{(3)}(0.29\lambda + 0.71).$$

The use of the root locus method yields to the poles $\{-3.3, -3.3, -5.4\}$ for $\Gamma_{(3)} = -12.56$. For these values we get the Luenberger observer defined by

$$\begin{aligned} G &= \begin{bmatrix} 83.8 & 319.8 & 774.4 & 1381.3 & -49.9 \\ 74.1 & 140 & 368 & 671.6 & -5.1 \\ 13.6 & 14 & 42 & 79 & 2 \end{bmatrix} B, \\ &= \begin{bmatrix} -49.92 \\ -5.12 \\ 2 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 & -58.88 \\ 1 & 0 & -46.55 \\ 0 & 1 & -12 \end{bmatrix}, \end{aligned}$$

$$H = \begin{bmatrix} -4036.1 \\ -1844.8 \\ -198.9 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad V = -12.56.$$

In Fig. 1 the simulation results are displayed showing the performances of the second-order and the third-order designed observers.

4. Conclusion

We have proposed a new and direct design procedure of a minimal Luenberger observer for a single linear functional. Our algorithm is based on linear algebraic operations in a state space setting. With respect to other procedures the design procedure does not require the solution of a Sylvester equation. Moreover, the proposed solution exhibits design parameters for the candidate observer to achieve asymptotic stability or pole placement when some poles are fixed at the outset. Let us mention that we do not suppose any canonical form either for the system or for the observer. The proposed constructive procedure is simpler than the Trinh procedure (Trinh et al., 2006) which is based on the resolution of the Sylvester equation with the Duan method.

The proposed design principle can be extended twofold. On the one hand to time-varying linear systems. On the other hand for the multifunctional case. Such developments are under investigation and will be the subjects of future works. Nevertheless it can been shown that in the multifunctional case the existence condition of a minimum observer can be given by an obvious extension of the Theorem 1.

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